Random Graphs
Exercise Sheet 4

Question 1. Let $p = \frac{c}{n}$ with $c > 1$. Recall that with high probability there is a unique 'giant' component of $G_{n,p}$ of size $(1 + o(1))\beta_c n$ for some β_c. How many edges are in the giant component?

(It may be easier to consider the $G_{n,m}$ model)

Question 2. Let $\alpha, \delta > 0$, let P be a path of length αn with $V(P) \subseteq [n]$ and let $p = \frac{\delta}{n}$. Suppose we split P into 5 paths $P_1, P_2, \ldots, P_5$ of equal length, show that whp $G(n,p)$ contains an edge between each pair of paths.

Deduce that if $p = \frac{c}{n}$ with $c > 1$ then whp $G(n,p)$ is non-planar.

Question 3. The *k-core* of a graph G is the maximal induced subgraph of G with minimum degree at least k.

Show that if c is large enough then with high probability $G_{n,\frac{c}{n}}$ has a non-empty k-core. Show further that it is linear in size.

Question 4. Let $D(n,p)$ be a random digraph formed by taking, for each ordered pair $(x, y) \in [n]^2$ and edge from x to y independently with probability p.

Let $p = \frac{c}{n}$ with $c > 1$. Show that whp $D(n,p)$ contains a directed cycle of length $\Theta(n)$.

Question 5. Show if every two disjoint vertex sets of size m contain an edge between them then G contains a path of length $n - 2m$.

Show that for every $\varepsilon > 0$ there is a $c > 0$ such that if $p = \frac{c}{n}$ then with high probability $G_{n,p}$ contains a path of length $(1 - \varepsilon)n$.

Question 6. Using the previous question, show that for sufficiently large C and $p = \frac{C^2}{n}$ with high probability every 2-colouring of the edges of $G_{n,p}$ contains a monochromatic path of length at least $\frac{n}{C}$.